

Scaling limit theorems for the κ -transient random walk in random and non-random environment ¹

Wenming Hong² Hui Yang³
(Beijing Normal University)

Abstract

Kesten et al.(1975) proved the stable law for the transient RWRE (here we refer it as the κ -transient RWRE). After that, some similar interesting properties have also been revealed for its continuous counterpart, the diffusion proces in a Brownian environment with drift κ . In the present paper we will investigate the connections between these two kind of models, i.e., we will construct a sequence of the κ -transient RWREs and prove it convergence to the diffusion proces in a Brownian environment with drift κ by proper scaling. To this end, we need a counterpart convergence for the κ -transient random walk in non-random environment, which is interesting itself.

Keywords: Random walk, Random environment, Diffusion process, Brownian motion with drift.

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1 Introduction

Let $\omega := \{\omega_i\}_{i \in \mathbb{Z}}$ be a sequence of independent and identically distributed random variables taking values in $(0, 1)$, which is labeled as “random environment”. For any realization of ω , we define a Markov chain $\{Z_n\}_{n \geq 0}$ with $Z_0 = 0$ and for any $n \geq 0$ and $i \in \mathbb{Z}$,

$$\mathbb{P}(Z_{n+1} = j \mid Z_n = i; \omega) = \begin{cases} \omega_i, & \text{if } j = i + 1, \\ 1 - \omega_i, & \text{if } j = i - 1, \\ 0, & \text{otherwise,} \end{cases} \quad (1.1)$$

The process $\{Z_n\}_{n \geq 0}$ is a so-called random walk in random environment (RWRE, for simplicity), we refer the readers to Zeitouni ([18], 2004) for more details. We denote P as the law of the

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² School of Mathematical Sciences & Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R. China. Email: wmhong@bnu.edu.cn

³ School of Mathematical Sciences & Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R. China. Email: yanghui2011@mail.bnu.edu.cn

environment, P_ω and \mathbb{P} as the quenched and annealed probability of the RWRE respectively, and E , E_ω and \mathbb{E} as the corresponding expectation.

In the famous paper, Kesten et al. ([11], 1975) proved the stable law for the transient RWRE. The main assumptions on the random environment ω is, (denote $\rho_i = \frac{1-\omega_i}{\omega_i}$, $i \in \mathbb{Z}$)

$$\begin{cases} \{\omega_i\}_{i \in \mathbb{Z}} \text{ is a sequence of i.i.d. random variables taking values in } (0, 1), \\ E(\log \rho_0) < 0, \\ \exists \kappa > 0, \text{ s.t. } E\rho_0^\kappa = 1. \end{cases} \quad (1.2)$$

The condition $E(\log \rho_0) < 0$ indicates the RWRE is transient, and intuitively the κ in $E\rho_0^\kappa = 1$ is a “measurement” for the transient in some sense, and we call it the κ -transient RWRE. In addition of the stable law (see [11], 1975), many other interesting results on this κ -transient RWRE has been obtained, for example, the large deviation principles have been proved by Greven and den Hollander ([8], 1994), Dembo et al ([4], 1996) and Comets et al ([3], 2000); see also Enriquez et al ([6], 2009); etc..

On the other hand, as a continuous version of the RWRE, the diffusion process with random potential has been considered in the literature. Let $V = (V(x), x \in \mathbb{R})$ be a stochastic process defined on \mathbb{R} with $V(0) = 0$. Define $X_V = (X_V(t), t \geq 0)$ a diffusion process in random potential V as an informal solution of the stochastic differential equation

$$\begin{cases} dX_V(t) = dB(t) - \frac{1}{2}V'(X_V(t))dt, \\ X_V(0) = 0. \end{cases} \quad (1.3)$$

where $(B(t), t \geq 0)$ is a standard Brownian motion independent of V . Strictly speaking, instead of writing the formal derivation of V , the process X_V can be considered as a diffusion process with generator

$$\frac{1}{2}e^{V(x)} \frac{d}{dx} \left(e^{-V(x)} \frac{d}{dx} \right).$$

It is called *diffusion proces in a Brownian environment with drift* $\kappa \geq 0$ if

$$V(x) = \sigma \cdot W(x) - \frac{\kappa}{2}x, \quad x \in \mathbb{R}, \quad (1.4)$$

where $\{W(x); x \in \mathbb{R}\}$ is a two-sided Brownian motion with $W(0) = 0$.

(1) When $\kappa = 0$, X_V is also called *Brox* diffusion, which was discussed by (Schumacher [13] and Brox [2]). They showed that X_V exhibits the same asymptotic behavior as Sinai’s random walk (i.e., $E(\log \rho_0) = 0$), and the *Brox* diffusion can be viewed as a continuous version of the Sinai’s random walk. Indeed, Seignourel ([14], 2000) proved that a sequence of Sinai’s random walk convergence to the Brox’s diffusion by scaling.

(2) When $\kappa > 0$, X_V exhibits the similar asymptotic behavior as the κ -transient RWRE (Kesten et al. [11], 1975) such as the stable law (Kawazu and Tanaka [10], and Tanaka [17]) and the large deviations (Hu et al. [9] and Taleb [16]), etc., and the diffusion X_V can be viewed as a continuous version of the κ -transient RWRE in some sense. It is naturally to ask the corresponding scaling limit relations, similar as for the situation $\kappa = 0$ by Seignourel ([14], 2000).

The purpose of the present paper is to construct a sequence of the κ -transient RWREs and to prove it convergence to the diffusion proces in a Brownian environment with drift κ by proper scaling. To this end, we need a counterpart convergence for the random walk in non-random environment, which is interesting itself.

2 β -transient random walk in non-random environment

$\{X_i, i \in \mathbb{N}^+\}$ is a sequence of i.i.d. random variables taking values in \mathbb{R} whose distribution denoted by μ , and satisfies the following assumptions:

$$\begin{cases} 1. EX_1 < 0, \\ 2. \exists \beta > 0 \text{ s.t. } Ee^{\beta X_1} = 1, \\ 3. \mu \text{ has compact support.} \end{cases} \quad (2.1)$$

Define a random walk $\{S_n\}_{n \geq 0}$ with $S_0 = 0$ and for $n \geq 1$,

$$S_n = \sum_{i=1}^n X_i.$$

Conditions in (2.1) states that the random work is transient to $-\infty$ with non-zero speed, and the value of β “measure” the transient of the random walk in some sense. It is worthwhile to point out that, by the Donsker’s invariant principle with the usual schedule, one can not obtain the limit process as the Brownian motion with drift. To this end, we must construct a sequence of random walks with “measure change”. Borisov et al ([1]) considered the transient simplest random walk case. Here we consider the general situation under the condition (2.1), with the “transient measurement” β . A key step is to find the suitable “measure change” as the following,

♣ *step 1. Define $\mu^{(m)}$*

Starting from μ , which satisfies the condition (2.1). For each $m \geq 1$, $\mu^{(m)}$, is defined as following, for any continuous bounded function φ ,

$$\begin{aligned} \int_{\mathbb{R}} \varphi(x) \mu^{(m)}(dx) = & c^{-1} \int_0^\infty \mu(dv) \int_{-\infty}^0 \mu(du) (e^{\beta v} - e^{\beta u}). \\ & \left[\left(\frac{-v}{u-v} - \frac{\beta}{2\sqrt{m}(u-v)} \right) \varphi(u) + \left(\frac{u}{u-v} + \frac{\beta}{2\sqrt{m}(u-v)} \right) \varphi(v) \right], \end{aligned} \quad (2.2)$$

where $c := \int_0^\infty (e^{\beta x} - 1) \mu(dx) = \int_{-\infty}^0 -(e^{\beta x} - 1) \mu(dx) > 0$. Write

$$\sigma^2 = -c^{-1} \int_0^\infty \mu(dv) \int_{-\infty}^0 \mu(du) (e^{\beta v} - e^{\beta u}) uv. \quad (2.3)$$

By condition 3 in (2.1), we know $0 < \sigma^2 < \infty$. It is easy to check that $\mu^{(m)}$ is a probability measure for each $m \geq 1$.

♣ *step 2. Define $\{X_i^{(m)}, i \in \mathbb{N}^+\}$*

For each $m \geq 1$, define a sequence of i.i.d. random variables $\{X_i^{(m)}, i \in \mathbb{N}^+\}$, whose distribution is given by $\mu^{(m)}$, which is defined in (2.2). The corresponding random walk $\{S_n^{(m)}, n \geq 0\}$ is defined as $S_0^{(m)} = 0$, and

$$S_n^{(m)} = \sum_{i=1}^n X_i^{(m)}.$$

Proposition 2.1. *Under the assumption (2.1), we have as $m \rightarrow \infty$,*

$$\left\{ \frac{1}{\sqrt{m}} S_{[mt]}^{(m)}, t \in \mathbb{R}^+ \right\} \longrightarrow \left\{ H(t) = \sigma \cdot B(t) - \frac{\beta}{2} t, t \in \mathbb{R}^+ \right\} \quad (2.4)$$

in distribution in $\mathcal{D}[0, +\infty)$, Where σ is defined in (2.3) and $(B(t), t \geq 0)$ is a standard Brownian motion.

Remark 2.1. *If $\{S_n\}_{n \geq 0}$ is a simple random walk and satisfies the conditions (2.1), then μ the distribution of X_i is,*

$$\begin{aligned} \mu(1) &= \frac{1}{e^\beta + 1}, \\ \mu(-1) &= \frac{e^\beta}{e^\beta + 1}. \end{aligned}$$

It is easy to calculate from (2.2),

$$\begin{aligned} \mu^{(m)}(1) &= \frac{1}{2} + \frac{\beta}{4\sqrt{m}}, \\ \mu^{(m)}(-1) &= \frac{1}{2} - \frac{\beta}{4\sqrt{m}}, \end{aligned}$$

which is exactly the random walks constructed in the proof of Theorem 2 in [1]. For the general situation, the construction of the $\mu^{(m)}$ in (2.2) is stimulated by the Skorokhod's representation theorem (see for example page 399 in [5]).

Proof of Proposition 2.1. Write

$$H^{(m)}(t) = \begin{cases} \frac{1}{\sqrt{m}} S_k^{(m)}, & \text{if } t = \frac{k}{m}, k \in \mathbb{N}, \\ \text{linear on } [\frac{k}{m}, \frac{k+1}{m}], & \text{for } k \in \mathbb{N}. \end{cases} \quad (2.5)$$

it is enough to prove the weak convergence of $H^{(m)}(\cdot)$ to $H(\cdot)$ in $\mathcal{C}[0, \infty)$. To this end, we just need to illustrate the finite dimensional distributions of $H^{(m)}(\cdot)$ converge to the corresponding finite-dimensional distributions of $H(\cdot)$ as $m \rightarrow \infty$ and the relatively compact of $\{H^{(m)}(\cdot)\}_{m \geq 1}$ (Theorem 7.8 of Chapter 3 in ([7])).

(1) *convergence of the finite dimensional distributions of $H^{(m)}(\cdot)$.*

By Theorem 6.5 of Chapter 1 and Corollary 8.5 of Chapter 4 in ([7]), we just need to verify that,

$$mA_m f \rightarrow Lf \quad (2.6)$$

uniformly on compact subsets of \mathbb{R} , where $L = \frac{1}{2}\sigma^2 \frac{d^2}{(dx)^2} + (-\frac{\beta}{2})\frac{d}{dx}$ is the generator of $H(\cdot)$ and

$$A_m f(x) = \int (f(y) - f(x)) \Pi_m(x, dy),$$

for all $f \in C_0^\infty(\mathbb{R})$. $\Pi_m(x, \Gamma)$ is transition probability of function $H^{(m)}$, defined as following,

$$\begin{aligned} \Pi_m(x, \Gamma) &:= P\left(H^{(m)}\left(\frac{k+1}{m}\right) \in \Gamma \mid H^{(m)}\left(\frac{k}{m}\right) = x\right) \\ &= P\left(\frac{1}{\sqrt{m}}S_{k+1}^{(m)} \in \Gamma \mid \frac{1}{\sqrt{m}}S_k^{(m)} = x\right) \\ &= P\left(\left(x + \frac{1}{\sqrt{m}}X_{k+1}^{(m)}\right) \in \Gamma\right) = \int_{x + \frac{1}{\sqrt{m}}z \in \Gamma} \mu^{(m)}(dz). \end{aligned} \quad (2.7)$$

To prove (2.6), it is enough to check the following three conditions (2.8), (2.9) and (2.10) (by Lemma 11.2.1 ([15]))

$$\lim_{m \rightarrow \infty} \sup_{|x| \leq R} |a_m(x) - \sigma^2| = 0, \quad (2.8)$$

$$\lim_{m \rightarrow \infty} \sup_{|x| \leq R} |b_m(x) - (-\frac{\beta}{2})| = 0, \quad (2.9)$$

$$\lim_{m \rightarrow \infty} \sup_{|x| \leq R} \Delta_m^\varepsilon = 0, \quad \varepsilon > 0. \quad (2.10)$$

for all $R > 0$, where

$$\begin{aligned} a_m(x) &= m \int_{|y-x| \leq 1} (y-x)^2 \Pi_m(x, dy), \\ b_m(x) &= m \int_{|y-x| \leq 1} (y-x) \Pi_m(x, dy), \\ \Delta_m^\varepsilon &= m \Pi_m(x, \mathbb{R} \setminus B(x, \varepsilon)), \quad \varepsilon > 0. \end{aligned}$$

Here, for large enough m , by the compact support of the distribution μ and the expression of $\Pi_m(x, \Gamma)$ in (2.7), it is easy to calculate that

$$\begin{aligned} a_m(x) &= m \int_{|y-x| \leq 1} (y-x)^2 \Pi_m(x, dy), \\ &= m \int_{|\frac{1}{\sqrt{m}}z| \leq 1} \left(\frac{1}{\sqrt{m}}z\right)^2 \mu^{(m)}(dz), \\ &= \int_{\mathbb{R}} z^2 \mu^{(m)}(dz), \\ &= c^{-1} \int_0^\infty \mu(dv) \int_{-\infty}^0 \mu(du) (e^{\beta v} - e^{\beta u}) \left(-uv - \frac{\beta(u+v)}{2\sqrt{m}}\right), \end{aligned}$$

the last equality is followed from the construction of $\mu^{(m)}$ in (2.2). Similarly, we can calculate that

$$b_m(x) = m \int_{|y-x| \leq 1} (y-x) \Pi_m(x, dy),$$

$$\begin{aligned}
&= m \int_{\frac{1}{\sqrt{m}}z \leq 1} \frac{1}{\sqrt{m}} z \mu^{(m)}(dz), \\
&= -\frac{\beta}{2},
\end{aligned}$$

and

$$\begin{aligned}
\Delta_m^\varepsilon &= m \Pi_m(x, \mathbb{R} \setminus B(x, \varepsilon)) = \int_{x + \frac{1}{\sqrt{m}}z \in \mathbb{R} \setminus B(x, \varepsilon)} \mu^{(m)}(dz), \\
&= \int_{\frac{1}{\sqrt{m}}z \in \mathbb{R} \setminus B(0, \varepsilon)} \mu^{(m)}(dz) = \int_{|\frac{1}{\sqrt{m}}z| > \varepsilon} \mu^{(m)}(dz) = 0.
\end{aligned}$$

obviously, which satisfy the conditions (2.8), (2.9), (2.10).

(2) *relatively compact of $\{H^{(m)}(\cdot)\}_{m \geq 1}$.*

With (2.6) in hand, by Corollary 4.3 of Chapter 4 in ([12]), we need only to verify the following condition, that is, for all $\epsilon, T > 0$ there exist compact sets $K_{\epsilon, T} \subset \mathbb{R}$ such that

$$\inf_m P\{H^{(m)}(t) \in K_{\epsilon, T} \text{ for all } t \leq T\} \geq 1 - \epsilon. \quad (2.11)$$

At first recall that for each $m \geq 1$, $X_i^{(m)}, i \geq 1$ are i.i.d. random variables, determined by $\mu^{(m)}$ in (2.2). It is easy to calculate the moments of $X_i^{(m)}$,

$$\begin{aligned}
EX_i^{(m)} &= c^{-1} \int_0^\infty \mu(dv) \int_{-\infty}^0 \mu(du) (e^{\beta v} - e^{\beta u}). \\
&\quad \left[\left(\frac{-v}{u-v} - \frac{\beta}{2\sqrt{m}(u-v)} \right) u + \left(\frac{u}{u-v} + \frac{\beta}{2\sqrt{m}(u-v)} \right) v \right] \\
&= -\frac{\beta}{2\sqrt{m}},
\end{aligned}$$

for each $i \geq 1$, and similarly

$$D(X_i^{(m)}) = c^{-1} \int_0^\infty dF(v) \int_{-\infty}^0 dF(u) (e^{\beta v} - e^{\beta u}) (-uv - \frac{\beta(u+v)}{2\sqrt{m}}) - \frac{\beta^2}{4m}. \quad (2.12)$$

Let $\eta_i^{(m)} := (X_i^{(m)} - EX_i^{(m)})/\sqrt{m} = (X_i^{(m)} + \frac{\beta}{2\sqrt{m}})/\sqrt{m}$ and $M_n^{(m)} = \sum_{i=1}^n \eta_i^{(m)}$, then for each $m \geq 1$, $\eta_i^{(m)}, i \geq 1$ are i.i.d. random variables with $E\eta_i^{(m)} = 0$, $Var\eta_i^{(m)} = \frac{DX_i^{(m)}}{m}$.

$$\begin{aligned}
P \left\{ \sup_{0 \leq x \leq T} |H^{(m)}(x)| \geq \lambda \right\} &= P \left\{ \sup_{1 \leq n \leq \lceil mT \rceil} \frac{1}{\sqrt{m}} \left| \sum_{i=1}^n X_i^{(m)} \right| \geq \lambda \right\}, \\
&= P \left\{ \sup_{1 \leq n \leq \lceil mT \rceil} \left| \frac{1}{\sqrt{m}} \sum_{i=1}^n (X_i^{(m)} + \frac{\beta}{2\sqrt{m}}) - \frac{n\beta}{2m} \right| \geq \lambda \right\} \\
&\leq P \left\{ \sup_{1 \leq n \leq \lceil mT \rceil} |M_n^{(m)}| \geq \lambda \right\}, \quad (2.13)
\end{aligned}$$

By Kolmogorov's maximal inequality (page 61, [5]), (2.13) can be continuous

$$P \left\{ \sup_{0 \leq x \leq T} |H^{(m)}(x)| \geq \lambda \right\} \leq \frac{\text{Var} M_{[mT]}^{(m)}}{\lambda^2} = \frac{[mT] \frac{DX_i^{(m)}}{m}}{\lambda^2} \leq \frac{(T+1)DX_i^{(m)}}{\lambda^2} \leq \frac{2TC}{\lambda^2},$$

where $[mT]$ denotes the smallest integer greater than mT . The last inequality is because the sequence of $\{DX_i^{(m)}\}$ is convergent as $m \rightarrow \infty$ (see (2.12)), and then there is a constant $C > 0$ such that $DX_i^{(m)} \leq C$. Now we choose λ large enough such that $\frac{2TC}{\lambda^2} < \epsilon$, as a consequence (2.11) is proved if we take $K_{\epsilon,T} = [-\lambda - 1, \lambda + 1]$. \square

3 κ -transient random walk in random environment

Let us start from a κ -transient RWRE $\{Z_n\}_{n \geq 0}$ with the i.i.d. environment series $\{\omega_i, i \in \mathbb{Z}\}$ satisfying the condition (1.2), and assume the marginal distribution of ω_i is ν . Recall $\rho_i = \frac{1-\omega_i}{\omega_i}$, $i \in \mathbb{Z}$; we write the distribution of $\log \rho_0$ as π . Now we can construct a sequence of κ -transient RWRE as the following four steps,

♣ *step 1. Define $\pi^{(m)}$.*

We will construct a sequence of $\pi^{(m)}$ by (2.2) replacing $\mu = \pi$ and $\beta = \kappa$ respectively, i.e., for any continuous bounded function φ ,

$$\int_{\mathbb{R}} \varphi(x) \pi^{(m)}(dx) = c^{-1} \int_0^\infty \pi(dv) \int_{-\infty}^0 \pi(du) (e^{\kappa v} - e^{\kappa u}).$$

$$\left[\left(\frac{-v}{u-v} - \frac{\kappa}{2\sqrt{m}(u-v)} \right) \varphi(u) + \left(\frac{u}{u-v} + \frac{\kappa}{2\sqrt{m}(u-v)} \right) \varphi(v) \right], \quad (3.1)$$

where $c := \int_0^\infty (e^{\kappa x} - 1) \pi(dx) = \int_{-\infty}^0 -(e^{\kappa x} - 1) \pi(dx) > 0$.

♣ *step 2. Define $\{\delta_i^{(m)}\}_{i \in \mathbb{Z}}$.*

Let $\{\delta_i^{(m)}\}_{i \in \mathbb{Z}}$ be a sequence of i.i.d. random variables with distribution $\pi^{(m)}$, which is defined in (3.1).

♣ *step 3. Define "random environment" $\omega_i^{(m)}$.* Let

$$\omega_i^{(m)} := \left(1 + e^{\delta_i^{(m)}/\sqrt{m}} \right)^{-1}. \quad (3.2)$$

♣ *step 4. Define "random walk in random environment" $Z^{(m)}$.*

Denote $Z^{(m)} = \{Z_n^{(m)}\}_{n \geq 0}$ as the random walk associated with the random environment $\omega^{(m)} = \{\omega_i^{(m)}\}_{i \in \mathbb{Z}}$, i.e.,

$$\mathbb{P} \left(Z_{n+1}^{(m)} = j \mid Z_n^{(m)} = i; \omega^{(m)} \right) = \begin{cases} \omega_i^{(m)}, & \text{if } j = i + 1, \\ 1 - \omega_i^{(m)}, & \text{if } j = i - 1, \\ 0, & \text{otherwise.} \end{cases} \quad (3.3)$$

From the above construction, we have

Proposition 3.1. *For each $m \geq 1$, the RWREs $Z^{(m)} = \{Z_n^{(m)}\}_{n \geq 0}$ are κ_m -transient RWRE, i.e., each of them satisfies condition (1.2) with some $\kappa_m > 0$.*

Proof For each $m \geq 1$, denote $\rho_i^{(m)} = \frac{1-\omega_i^{(m)}}{\omega_i^{(m)}}$, then, we have

$$E \log \rho_0^{(m)} = -\frac{\kappa}{2m} < 0,$$

so $Z^{(m)}$ is transient to $+\infty$. In addition, by the convex of the function of $f(s) = E \exp s \log \rho_0^{(m)}$, $s \geq 0$ and $f'(0) = E \log \rho_0^{(m)} < 0$, we know that $\exists \kappa_m > 0$, s.t. $E(\rho_0^{(m)})^{\kappa_m} = 1$. So, for every $m \geq 1$, RWRE $Z^{(m)}$ is an κ_m -transient RWRE, i.e., it satisfies condition (1.2), a Kesten's type RWRE. \square

Theorem 3.1. *Under the assumptions (1.2) and uniformly elliptic, i.e. $\exists 0 < \varepsilon < 1/2$ such that $P(\omega_0 \in [\varepsilon, 1 - \varepsilon]) = 1$; $Z^{(m)}$ be defined in (3.3). Then, as $m \rightarrow \infty$*

$$\left\{ \frac{1}{m} Z_{[m^2 t]}^{(m)}, t \geq 0 \right\} \longrightarrow \{X_V(t), t \geq 0\} \quad (3.4)$$

in distribution in $\mathcal{D}[0, \infty)$. Where $X_V = (X_V(t), t \geq 0)$ a diffusion process in random potential $V(x) = \sigma \cdot W(x) - \frac{\kappa}{2}x$ (defined in (1.3) and (1.4) respectively), and σ^2 defined in (2.3) replacing $\mu = \pi$ and $\beta = \kappa$.

Remark 3.1. *Recall Seignourel [14] proved that the scaling limit of Sinai's random walk random environment (i.e. $E \log \rho_0 = 0$) is Brox's diffusion (i.e. $V = \text{Wiener process}$).*

Theorem A (Seignourel, [14], 2000) Assume $E \log \rho_0 = 0$, and denote $\sigma^2 := E(\log \rho_0)^2$. For every $m \geq 1$ and for every $i \in \mathbb{Z}$, let

$$\omega_i^{(m)} := (1 + (\rho_i)^{\frac{1}{\sqrt{m}}})^{-1} = \left(1 + e^{\log \rho_i / \sqrt{m}}\right)^{-1}. \quad (3.5)$$

Then, as $m \rightarrow \infty$

$$\left\{ \frac{1}{m} Z_{[m^2 t]}^{(m)}, t \geq 0 \right\} \longrightarrow \{X_V(t), t \geq 0\}$$

in distribution in $\mathcal{D}[0, \infty)$, where $\{X_V(t), t \geq 0\}$ is the diffusion process with random potential $V = \sigma \cdot \text{Brownian motion}$.

Comparing the “environment” definition in (3.2) and (3.5), our “measure change” related to $m \geq 1$ is through the $\delta_i^{(m)}$ in (3.2). \square

Proof of Theorem 3.1 We can follows the proof of the Theorem 1 in [14] (Seignourel, 2000) almost line by line, except that the $W^m(x)$ in Theorem 12 of [14] is now that

$$V^{(m)}(x) = \begin{cases} \frac{1}{\sqrt{m}} \sum_{i=1}^{[mx]} \delta_i^{(m)}, & \text{if } [mx] \geq 1, \\ 0, & \text{if } 0 \leq [mx] \leq 1, \\ -\frac{1}{\sqrt{m}} \sum_{i=1}^{[mx]} \delta_i^{(m)}, & \text{if } [mx] < 0, \end{cases} \quad (3.6)$$

where $\{\delta_i^{(m)}\}_{i \in \mathbb{Z}}$ be a sequence of i.i.d. random variables with distribution $\pi^{(m)}$, which is defined in (3.1). By the definition of $\omega_i^{(m)}$ in (3.2), actually we know with $\rho_i^{(m)} = \frac{1-\omega_i^{(m)}}{\omega_i^{(m)}}$,

$$\delta_i^{(m)} = \sqrt{m} \log \rho_i^{(m)}.$$

The proof will complete if we can verify the sequence of $V^{(m)}(x)$ convergence weakly to $V(x) = \sigma \cdot W(x) - \frac{\kappa}{2}x$ (defined in (1.4)).

To this end, we need only to take $X_i^{(m)} := \delta_i^{(m)}$ in Proposition 2.1; and by the construction of the sequences of $\{X_i^{(m)}\}$ and $\{\delta_i^{(m)}\}$, we know that $\{X_i\} := \{\log \rho_i\}$. The conditions (1.2) and uniformly elliptic conditions enable the sequences of $X_i^{(m)}$ (i.e., $\{\delta_i^{(m)}\}$) fulfill the conditions (2.1), as a consequence the sequence of $V^{(m)}(x)$ convergence weakly to $V(x) = \sigma \cdot W(x) - \frac{\kappa}{2}x$ by Proposition 2.1. \square

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